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Application of Fractional Power Series Method in Solving Fractional Differential Equations

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional power series, we use some examples to illustrate how to use fractional power series method to solve fractional differential equations. In fact, our results are generalizations of the results of ordinary differential equations.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional power series, fractional differential equations.**

I. INTRODUCTION

In a letter to L'Hospital in 1695, Leibniz proposed the possibility of generalizing classical differentiation to fractional order and asked what the result about $\frac{d^{1/2}x}{dx^{1/2}}$. After 124 years, Lacroix gave the right answer to this question for the first time that $\frac{d^{1/2}x}{dx^{1/2}} = \frac{2}{\sqrt{\pi}}x^{1/2}$. For a long time, due to the lack of practical application, fractional calculus has not been widely used. However, in the past few decades, fractional calculus has gained much attention as a result of its demonstrated applications in various fields of science and engineering such as physics, biology, electrical engineering, mechanics, elasticity, control theory, electronics, economics [1-12].

But the definition of fractional derivative is not unique, there are many useful definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov fractional derivative, Jumarie's modified R-L fractional derivative [13-17]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

Based on Jumarie's modified R-L fractional derivative, this paper gives some examples to illustrate how to use fractional power series method to solve fractional differential equations. A new multiplication of fractional power series plays an important role in this paper. In fact, our results are generalizations of ordinary differential equation results.

II. PRELIMINARIES

Firstly, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([18])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer p, we define $\left({}_{x_0}D_x^{\alpha}\right)^p[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the p-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([19])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Definition 2.3 ([20])**:** Let x, x_0 and a_k be real numbers for all k, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional power series at $x = x_0$.

In the following, we introduce a new multiplication of fractional power series.

Definition 2.4 ([21])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{7}$$

Definition 2.5 ([22])**:** Assume that $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes k}. \tag{8}$$

III. EXAMPLES

In this section, we use fractional power series method to solve some fractional differential equations.

Example 3.1: If $0 < \alpha \leq 1$. Find the particular solution of the following initial value problem of α-fractional differential equation:

$$\begin{cases} \left({}_0D_x^{\alpha}\right)^2[y_\alpha(x^\alpha)] - \frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha y_\alpha(x^\alpha) = 0, \\ y_\alpha(0) = 0, \quad \left({}_0D_x^{\alpha}\right)[y_\alpha(x^\alpha)](0) = 1 \end{cases}, \tag{9}$$

Solution Suppose that the particular solution is

$$\begin{aligned} y_\alpha(x^\alpha) & = \sum_{k=0}^{\infty} a_k \left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha k} \\ & = a_0 + a_1\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right) + a_2\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2} + \cdots + a_k\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha k} + \cdots. \end{aligned} \tag{10}$$

Since $y_\alpha(0) = 0$, it follows that $a_0 = 0$. In addition,

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$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] = \sum_{k=1}^{\infty} k\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-1)}$$

$$= a_1 + 2a_2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) + 3a_3\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2} + \cdots + k\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-1)} + \cdots. \tag{11}$$

Since $\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)](0) = 1$, we obtain $a_1 = 1$. Thus,

$$y_\alpha(x^\alpha) = \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) + a_2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2} + \cdots + a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k} + \cdots$$

$$= \frac{1}{\Gamma(\alpha+1)}x^\alpha + \sum_{k=2}^{\infty} a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}. \tag{12}$$

And

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] = 1 + 2a_2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) + 3a_3\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2} + \cdots + k\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-1)} + \cdots$$

$$= 1 + \sum_{k=2}^{\infty} k\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-1)}. \tag{13}$$

Furthermore,

$$\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)] = 2a_2 + 3\cdot 2a_3\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) + \cdots + k(k-1)\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-2)} + \cdots$$

$$= \sum_{k=2}^{\infty} k(k-1)\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-2)}. \tag{14}$$

Since $\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)] - \frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha y_\alpha(x^\alpha) = 0$, it follows that

$$\sum_{k=2}^{\infty} k(k-1)\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-2)} - \frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes_\alpha \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha + \sum_{k=2}^{\infty} a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}\right] = 0. \tag{15}$$

And hence,

$$\sum_{k=2}^{\infty} k(k-1)\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-2)} - \left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2} - \sum_{k=2}^{\infty} a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k+1)} = 0. \tag{16}$$

That is,

$$2a_2 + 3\cdot 2a_3\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right) + (4\cdot 3a_4 - 1)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2} + (5\cdot 4a_5 - a_2)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 3}$$

$$+(6\cdot 5a_6 - a_3)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 4} + \cdots + [(k+2)(k+1)\cdot a_{k+2} - a_{k-1}]\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k} + \cdots = 0. \tag{17}$$

Therefore,

$$a_2 = 0,\ a_3 = 0,\ a_4 = \frac{1}{4\cdot 3},\ a_5 = 0,\ a_6 = 0, \cdots. \tag{18}$$

Generally,

$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)} \tag{19}$$

for $k = 1,2,\cdots$.

Hence,

$$a_7 = \frac{a_4}{7\cdot 6} = \frac{1}{7\cdot 6\cdot 4\cdot 3},\ a_8 = \frac{a_5}{8\cdot 7} = 0,\ a_9 = \frac{a_6}{9\cdot 8} = 0,\ a_{10} = \frac{a_7}{10\cdot 9} = \frac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}, \cdots. \tag{20}$$

Generally,

$$a_{3m-1} = 0,\ a_{3m} = 0, \tag{21}$$

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and

$$a_{3m+1}=\frac{1}{(3m+1)\cdot 3m\cdot\cdots\cdot 10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} \tag{22}$$

for $m=1,2,\cdots$.

Thus, the particular solution of the initial value problem of α-fractional differential equation is

$$y_\alpha(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^\alpha+\frac{1}{4\cdot 3}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 4}+\frac{1}{7\cdot 6\cdot 4\cdot 3}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 7}+\frac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 10}+\cdots+$$

$$\frac{1}{(3m+1)\cdot 3m\cdot\cdots\cdot 10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (3m+1)}+\cdots$$

$$=\frac{1}{\Gamma(\alpha+1)}x^\alpha+\frac{1}{4\cdot 3}\cdot\frac{4!}{\Gamma(4\alpha+1)}x^{4\alpha}+\frac{1}{7\cdot 6\cdot 4\cdot 3}\cdot\frac{7!}{\Gamma(7\alpha+1)}x^{7\alpha}+\frac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}\cdot\frac{10!}{\Gamma(10\alpha+1)}x^{10\alpha}+\cdots+$$

$$\frac{1}{(3m+1)\cdot 3m\cdot\cdots\cdot 10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}\cdot\frac{(3m+1)!}{\Gamma((3m+1)\alpha+1)}x^{(3m+1)\alpha}+\cdots. \tag{23}$$

Example 3.2: Let $0<\alpha\leq 1$. Find the general solution of the following α-fractional differential equation:

$$\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+2\right)\otimes_\alpha y_\alpha(x^\alpha)=-2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}-2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{24}$$

Solution Let the general solution be

$$y_\alpha(x^\alpha)=\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}$$

$$=a_0+a_1\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)+a_2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}+\cdots+a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}+\cdots. \tag{25}$$

Then

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]=\sum_{k=0}^{\infty}k\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-1)}. \tag{26}$$

Since $\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+2\right)\otimes_\alpha y_\alpha(x^\alpha)=-2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}-2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha$, it follows that

$$0=\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes_\alpha\sum_{k=0}^{\infty}k\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k-1)}-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha+2\right)\otimes_\alpha\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}+2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}+2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha$$

$$=\sum_{k=0}^{\infty}k\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}-\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (k+1)}-2\cdot\sum_{k=0}^{\infty}a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}+2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}+2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha$$

$$=\sum_{k=0}^{\infty}(k-2)\cdot a_k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}-\sum_{k=1}^{\infty}a_{k-1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}+2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}+2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha$$

$$=-2a_0+\sum_{k=1}^{\infty}[(k-2)\cdot a_k-a_{k-1}]\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}+2\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}+2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{27}$$

Thus,

$$0=-2a_0+(-a_1-a_0+2)\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+(-a_1+2)\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}+\sum_{k=3}^{\infty}[(k-2)\cdot a_k-a_{k-1}]\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha k}. \tag{28}$$

Therefore,

$$a_0=0,\ a_1=2,\text{ and } a_k=\frac{a_{k-1}}{k-2}\text{ for } k=3,4,\cdots. \tag{29}$$

Hence,

$$a_3 = a_2,\ a_4 = \frac{a_3}{2} = \frac{a_2}{2!},\ a_5 = \frac{a_4}{3} = \frac{a_2}{3!},\ a_6 = \frac{a_5}{4} = \frac{a_2}{4!},\cdots. \tag{30}$$

Finally, we get the general solution is

$$y_\alpha(x^\alpha) = 2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha + a_2\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2} + a_2\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 3} + \frac{a_2}{2!}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 4} + \frac{a_2}{3!}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 5}$$

$$+\frac{a_2}{4!}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 6} + \cdots$$

$$= 2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha + a_2\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\otimes_\alpha\left[1 + \frac{1}{\Gamma(\alpha+1)}x^\alpha + \frac{1}{2!}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2} + \frac{1}{3!}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 3} + \cdots\right]$$

$$= 2\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha + a_2\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2}\otimes_\alpha E_\alpha(x^\alpha)\ , \tag{31}$$

where a_2 is a constant.

IV. CONCLUSION

Based on Jumarie's modified R-L fractional derivative, this paper provides some examples to illustrate how to use fractional power series method to solve fractional differential equations. A new multiplication of fractional power series plays an important role in this research. In fact, our results are generalizations of the results of ordinary differential equations. In the future, we will continue to use the fractional power series method to solve problems in fractional differential equations.

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